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CLASSIFICATION OF TWO-DIMENSIONAL ISENTROPIC GAS FLOWS OF DOUBLE-WAVE TYPE*

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A complete classification of two-dimensional isentropic gas flows of double-wave type (see /1 - 6/) when there is functional arbitrariness in the general solution of a Cauchy problem is given. Double waves were discussed earlier in the case of potential flows /1/. After replacing the potentiality conditions of the flow by the weaker condition regarding the rectilinearity of the contour lines (see /2/), a complete classification of two-dimensional isentropic gas flows of double-wave type with straight contour lines is given.

The gas-dynamic equations of a polytropic gas in the two-dimensional isentropic case can be written as

$$\frac{\partial v_i}{\partial t}+\sum_{k=1}^{2} v_k\frac{\partial v_i}{\partial x_k}+\frac{\partial\theta}{\partial x_i}=0,\quad i=1,2 \tag{1}$$

$$\frac{\partial\theta}{\partial t}+\sum_{k=1}^{2} v_k\frac{\partial\theta}{\partial x_k}-\varkappa\theta\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}\right)=0$$

$$\varkappa=\gamma-1,\ \theta=c^2/\varkappa$$

where (v_1, v_2) are the gas velocities, c is the velocity of light, and γ denotes the polytropic exponent of the gas.

Let V be a local neighbourhood of the point (v_1^c, v_2^c) in the space of the velocity locus. It is required that a travelling wave /1/ has, for a fixed function $\theta=\theta(v_1, v_2)$, $(v_1, v_2)\in V$, an arbitrariness regarding at least one function in the general solution of the Cauchy problem.

In the simplest case where $\theta_1^2+\theta_2^2=0$ ($\theta_i=\partial\theta/\partial v_i$, $\psi_i=\theta_i^2-\varkappa\theta$), it can be shown that the general solution has two arbitrary functions of one argument, and therefore we assume that $\theta_1^2+\theta_2^2\neq 0$. Then by rotating the coordinate system we can always contrive that in the new system, in a certain neighbourhood V of the point (v_1^c, v_2^c) of the velocity locus, the inequality

$$\theta_1\theta_2\psi_1\psi_2\neq 0 \tag{2}$$

will hold.

Further analysis is based on the fact that any consistent system of differential equations, after a finite number of extensions becomes an involutive system (see /7, 8/). If a system of differential equations is involutive, then the functional arbitrariness in the solution is determined by Cartan's characters which are connected in a definite way with the higher parametric determinants (see /7/). For solutions to exist, which have a functional arbitrariness, it is necessary that the rank of the matrix of the coefficients of the higher derivatives should not be equal to the number of all higher derivatives (under whatever assumptions).

On substituting $\theta=\theta(v_1, v_2)$ into Eq.(1) we obtain a system of quasilinear differential equations which is not involutive. It is necessary to extend this system when investigating it for consistency. Partially extending it, we change to the dependent variables $v_1, v_2, v_3=\partial v_1/\partial x_2-\partial v_2/\partial x_1$, and obtain an overdetermined system of five quasilinear first-order differential equations

$$S_i\equiv p_0^i+\sum_{k=1}^{2} v_k p_k^i+\theta_1 p_i^1+\theta_2 p_i^2=0,\quad i=1,2 \tag{3}$$

$$S_3\equiv p_0^3+\sum_{k=1}^{2} v_k p_k^3+v_3(p_1^1+p_2^2)=0$$

$$\Phi_1\equiv\psi_1 p_1^1+2\theta_1\theta_2 p_1^2+\psi_2 p_2^2+\theta_1\theta_2 v_3=0$$

$$\Phi_2\equiv p_2^1-p_1^2-v_3=0$$

$$(p_j^i=\partial v_i/\partial x_j,\ i=1,2,3;\ j=0,1,2;\ x_0\equiv t)$$

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Investigating this system for consistency, we obtain one more first-order equation

$$D_0\Phi_1 + \sum_{k=1}^{2}(v_k + \theta_k)D_k\Phi_1 + \theta_2(\theta_1^2 + \varkappa\theta)D_1\Phi_2 +$$
$$\theta_1\psi_2 D_2\Phi_2 - \psi_1 D_1 S_1 - 2\theta_1\theta_2 D_1 S_2 - \psi_2 D_2 S_2 =$$
$$\varkappa\theta(\theta_1 p_2^3 - \theta_2 p_1^3) + \sum_{i,k=1}^{2} b_{ik} p_i^i p_k^k + v_3(a_1 p_1^1 + a_2 p_2^2 + v_3 a_3) = 0 \tag{4}$$

where D_i $(i = 0, 1, 2)$ are total derivatives with respect to x_i, and the coefficients b_{ik}, a_j $(i, k = 1, 2;\ j = 1, 2, 3)$ depend on v_1 and v_2 only. Here we do not give the forms of b_{ik} and a_j since they are somewhat cumbersome.

Thus, the continuous solution of system (3) necessarily satisfies Eq.(4), and we can deduce the following. Firstly the vortex-free isentropic double waves ($v_3 = 0$) either have two arbitrary functions of one argument in the general solution of the Cauchy problem, or by the theorem on the reduction of double waves they are reduced to invariant solutions (see /9/). We therefore discuss below the turbulent solutions ($v_3 \neq 0$). Secondly, the maximal arbitrariness in the solution of a two-dimensional double isentropic wave for the specified function $\theta = \theta(v_1, v_2)$ is possible regarding three functions of one argument.

It can be shown that system (3),(4) is involutive only in the case when

$$\gamma = 2, \quad \theta = \frac{1}{2}\sum_{k=1}^{2}(v_k + c_k)^2 \tag{5}$$

(c_1, and c_2 are arbitrary constants). Here Eq.(4) takes the form

$$\theta(\theta_1 p_2^3 - \theta_2 p_1^3) + \frac{\theta^2 v_3}{\theta_1\theta_2}(p_1^1 - p_2^2) = 0$$

and the solution of system (3),(4) has three arbitary functions of one argument. Earlier, in classifying two-dimensional isentropic double waves with straight contour lines, such an equation for double waves was obtained in /2/, but the arbitrariness of the solution there indicated consisted of two functions of one argument, that is, the requirement of rectilinearity of the level reduces the arbitrariness to two functions of one argument. We will exclude the double-wave equation in the locus space from further discussion.

If conditions (5) are not satisfied, system (3), (4) is not involutive, and it is necessary to extend it when checking for consistency.

After extending the system by introducing dependent variables $v_4 = p_1^1$ and $v_5 = p_1^2$, and investigating the overdetermined system of ten quasilinear first-order differential equations in the dependent variables $v_1, v_2, \ldots v_5$, as was done for system (3), we obtain one more first-order equation

$$\sum_{i=3}^{5} Q_i p_1^i - Q_6 = 0 \tag{6}$$

In the above, the coefficients $Q_i = Q_i(v_1, \ldots, v_5)$ $(i = 3, 4, 5)$ are linear functions of v_3, v_4, and v_5. The form of these functions, being lengthy, is not given here.

It follows from the form of the overdetermined system of quasilinear differential equations in $v_1, v_2, \ldots, v_5$ that the parametric derivatives of higher order for the $(\alpha - 1)$-th extension can be $\partial^\alpha v_i/\partial x_1^\alpha (i = 3, 4, 5)$ only. Therefore, from among Cartan's characters responsible for the functional arbitrariness, only the first (σ_1) is non-zero, and at the same time the inequality $0 \leqslant \sigma_1 \leqslant 3$ holds.

If the system is in involution with Cartan's character then σ_1 functions of one argument are arbitary in the general solution of the Cauchy problem. Therefore the maximum possible arbitrariness regarding three functions of one argument is achieved only when

$$Q_i \equiv 0 \quad (i = 3, 4, 5) \tag{7}$$

otherwise $\sigma_1 < 3$. It follows from the form of the coefficients Q_i that the identities (7) are satisfied only when conditions (5) holds. Therefore, $Q_3^2 + Q_4^2 + Q_5^2 \not\equiv 0$.

After twice extending the overdetermined system of quasilinear differential equtions in $v_1, v_2, \ldots, v_5$, and compiling the linear combinations by excluding the main derivatives relatively to p_{111}^i $(i = 3, 4, 5)$, we obtain the following system of four linear algebraic equations:

$$\sum_{i=3}^{5} Q_i p_{111}^i = f_1, \quad Q_4 p_{111}^3 = f_2$$
$$((\theta_1^2 + \theta_2^2)Q_3 - \theta_1^2 Q_5)p_{111}^3 = f_3$$
$$\theta_2(-\psi_2 Q_3 + \theta_1^2 Q_5)p_{111}^3 + \theta_1\psi_1 Q_5 p_{111}^4 + \theta_1(-\psi_2 Q_4 + 2\theta_1\theta_2 Q_5)p_{111}^5 = f_4 \tag{8}$$

where the functions f_i $(i = 1, 2, 3, 4)$ depend on derivatives of order not higher than the second.

Because of the linearity of the extended systems with respect to higher derivatives, the process of constructing the linear combinations (8) can be performed in matrix form, which makes the mathematical operation much simpler.

It follows from (8) that for arbitrariness to exist in the general solution of the Cauchy problem it is necessary to satisfy the equation

$$(((\theta_1^2 + \theta_2^2)\, Q_3 - \theta_1^2 Q_5)^2 + Q_4^2)\,(\psi_2 Q_4^2 - 2\theta_1\theta_2 Q_4 Q_5 + \psi_1 Q_5^2) = 0 \tag{9}$$

which is an algebraic equation in $v_1, v_2, \ldots, v_5$. Extending (9) and performing an analogous investigation of the matrix consisting of the coefficients of the higher derivatives, we arrive at only two cases: either by the reduction theorem /9/ a double wave is reduced to an invariant solution, or a constant $\beta > 0$ exists such that $\theta_1^2 - \beta\theta_2^2 = 0$. (The lengthy intermediate operations are omitted: here we give the final result only.

In the latter case, we can achieve satisfaction of the equation $\theta_2 = 0$, that is $\theta = \theta(v_1)$, by rotating the coordinate system.

Consider the case when $\theta_2 = 0$. Substituting $\theta = \theta(v_1)$ into (3) and repeating the similar investigation regarding the existence of solutions of system (3) which possess functional arbitrariness, we arrive only at the case where $p_2^1 = 0$. Then, by the fourth equation of system (3) we have $p_{22}^2 = -D_2(\psi_1 p_1^1/\psi_2) = 0$, and therefore

$$v_2 = x_2 g_1(x_1, t) + g_2(x_1, t)$$

On substituting this expression into (3), and splitting S_1, we obtain a hyperbolic system of three equasilinear differential equations in $v_1(x_1, t), g_i(x_1, t)$ $(i = 1, 2)$

$$\begin{aligned} &\frac{\partial g_i}{\partial t} - v_1 \frac{\partial g_i}{\partial x_1} - g_1 g_i = 0 \quad (i = 1, 2) \\ &\frac{\partial v_1}{\partial t} + (v_1 - \theta')\frac{\partial v_1}{\partial x_1} = 0 \quad (\theta' \equiv \theta_1) \\ &\psi_1 \frac{\partial v_1}{\partial x_1} = \varkappa\theta g_1 \end{aligned} \tag{10}$$

This system is easily checked for consistency by cross differentiation. In the case of non-vortex flow $(g_1 = 0)$, we have either

$$(\theta')^2 - \varkappa\theta = 0, \quad \frac{\partial v_1}{\partial t} - (v_1 - \theta')\frac{\partial v_1}{\partial x_1} = 0, \quad \frac{\partial g_2}{\partial t} - v_1 \frac{\partial g_2}{\partial x_1} = 0 \tag{11}$$

that is the functions v_1, θ satisfy the equations of a simple wave as in the one-dimensional case, or

$$v_1 = \text{const}, \; v_2 = g_2(x_1 - v_1 t) \tag{12}$$

$(g_2(\zeta)$ is an arbitrary function). For vortex flow $(v_3 \neq 0)$, the function $\theta = \theta(v_1)$ should satisfy the ordinary third-order differential equation

$$\begin{aligned} &\psi_1 (1 - \theta'')\,(\psi_1 - \varkappa\theta\,(1 - \theta'')) - \theta'\,(\varkappa\theta\psi_1\theta''' - \\ &\varkappa\theta'\,(1 - \theta'')\,(\psi_1 - \theta\,(2\theta'' - \varkappa))) = 0 \end{aligned} \tag{13}$$

and the functions $v_1(x_1, t), g_i(x_1, t)$ $(i = 1, 2)$ satisfy the overdetermined system of first-order differential equations

$$\begin{aligned} &\frac{\partial g_i}{\partial t} - v_1 \frac{\partial g_i}{\partial x_1} - g_1 g_i = 0 \quad (i = 1, 2) \\ &\frac{\partial v_1}{\partial t} - v_1 \frac{\partial v_1}{\partial x_1} + \frac{\varkappa\theta}{\psi_1}\theta' g_1 = 0 \\ &\frac{\partial g_1}{\partial x_1} = \frac{1}{\theta'}\left(1 - (1 + \theta'')\frac{\varkappa\theta}{\psi_1}\right) g_1^2 \\ &\frac{\partial v_1}{\partial x_1} = -(v_1 + \theta')\frac{\varkappa\theta}{\psi_1} g_1 \end{aligned} \tag{14}$$

which is in involution and has an arbitrariness regarding one function of one argument (for example, $g_2(x_1, 0)$) in the general solution of the Cauchy problem.

Thus, we have presented the full classification of two-dimensional waves which have a functional arbitrariness in the general solution of Cauchy problems.

Theorem. The two-dimensional isentropic double waves possessing a functional arbitrariness in the general solution of the Cauchy problem for a given function $\theta = \theta(v_1, v_2)$ have the following forms only:

1) double waves reduced to invariant solutions;
2) double waves (5) which possess arbitrariness regarding three functions of one argument;
3) vortex-free double waves which possess arbitrariness regarding two functions of one argument;

4) double waves with $\theta(v_1)$, in which the arbitrariness is determined by one function of one argument $v_1 = v_1(x_1, t)$, $v_2 = x_2 g_1(x_1, t) + g_2(x_1, t)$, and the following holds: a) condition (11) or (12) for $g_1 = 0$, b) equations (13) and (14) for $g_1 \neq 0$.

The process by which this classification was established is a generalization of /1/, and it can be successfully used for other types of gas flow. For example, for vortex-free isentropic flows of a treble-type wave (see /10/) a solution with maximum possible arbitrariness regarding two functions of two arguments exists only when

$$\theta = c_0 + \sum_{i=1}^{3} \frac{1}{2}(c_i - v_i)^2 \tag{15}$$

Here the travelling wave with $x_i - c_i t = \Pi_i$ $(i = 1, 2, 3)$, where the function $\Pi(v_1, v_2, v_3)$ satisfies the equation

$$\psi_1 \begin{vmatrix} \Pi_{22} & \Pi_{23} \\ \Pi_{32} & \Pi_{33} \end{vmatrix} + \psi_2 \begin{vmatrix} \Pi_{11} & \Pi_{13} \\ \Pi_{31} & \Pi_{33} \end{vmatrix} + \psi_3 \begin{vmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{vmatrix} - \tag{16}$$

$$2\theta_1\theta_2 \begin{vmatrix} \Pi_{21} & \Pi_{23} \\ \Pi_{31} & \Pi_{33} \end{vmatrix} + 2\theta_1\theta_3 \begin{vmatrix} \Pi_{12} & \Pi_{13} \\ \Pi_{22} & \Pi_{23} \end{vmatrix} - 2\theta_2\theta_3 \begin{vmatrix} \Pi_{11} & \Pi_{13} \\ \Pi_{21} & \Pi_{23} \end{vmatrix} = 0$$

$$\left(c_i = \text{const}, \ \Pi_i = \frac{\partial \Pi}{\partial v_i}, \quad \Pi_{ij} = \frac{\partial^2 \Pi}{\partial v_i \partial v_j}, \quad i, j = 1, 2, 3\right)$$

will be the solution.

As remarked by A.F. Sidorov, after the change $x_i' = x_i - c_i t$ $(i = 1, 2, 3)$, a case simply of space potential stationary motions will be obtained in x_i' coordinates (all motions are treble waves). The representation (15) of θ will correspond to the Bernoulli integral, and equation (16) for $\Pi(v_1, v_2, v_3)$ will correspond to the equation for the velocity potential, transformed by the Legendre change.

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